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Analytical Mechanics Phys (252)

Lecture 10

Forces that Depend on Time

The Concept of Impulse

Forces of extremely short duration in time, such as those exerted by bodies undergoing collisions, are called impulsive forces.

If we confine our attention to one body, or particle, the differential equation of motion is

$$d(mv) = F\,dt.$$

Let us take the time integral over the interval t_1 to t_2, the time during which the force is considered to act, then we have

$$\Delta(mv) = \boxed{\int_{t_1}^{t_2} F\,dt = P}$$

The Impulse

The *time integral of the force* is ***the impulse***. It is usually denoted by the symbol ***P***.

Note:

1- The ***work*** is equal to the change in the energy of the particle.

$$\Delta T = \int_{x_1}^{x_2} F\,dx = W$$

2- The ***impulse*** is equal to the change in the momentum of the particle.

$$\Delta(mv) = \int_{t_1}^{t_2} F\,dt = P$$

Forces that Depend on Velocity

Fluid Resistance and Terminal Velocity

The force acts on a body is often a function of its velocity.
For example, the viscous resistance exerted on a body moving through a fluid depends on its velocity. In such case, the differential equation of motion may be written in either of the two forms

$$F_0 + F(v) = m\frac{dv}{dt}$$

$$F_0 + F(v) = mv\frac{dv}{dx}$$

Here F_0 is any constant force that does not depend on v.

Since, $F(v)$ is a complex function and must be found through experimental measurements, it can be replaced by the following approximation :

$$\boxed{\begin{aligned} F(v) &= -c_1 v - c_2 v|v| \\ F(v) &= -v(c_1 + c_2|v|) \end{aligned}}$$

where c_1 and c_2 are constants whose values depend on the size and shape of the body.

Linear or Quadratic ?

For spheres in air,

$$c_1 = 1.55 \times 10^{-4} D \qquad \& \qquad c_2 = 0.22\, D^2$$

where *D* is the diameter of the sphere in ***meters***.
For *small v* the ***linear*** term in *F(v)* can be used ,
while the ***quadratic*** term dominates at *large v*.

To decide whether the case is linear or quadratic, the ratio of the latter to the former usually used;

$$\frac{c_2 v|v|}{c_1 v} = \frac{0.22\, v|v| D^2}{1.55 \times 10^{-4}\, vD} = 1.4 \times 10^3 |v| D$$

If the value of *v* will make the ratio **exceeds 1** then it is a quadratic case, otherwise, it is a linear one.

Horizontal Motion through a Fluid

Linear Resistance

(Exp.2.4.1)

Suppose a block is projected with initial velocity v_0 on a smooth horizontal surface and that there is air resistance such that the ***linear term*** dominates.

Hence, $F_0 = 0$, and $F(v) = -c_1 v$.

The differential equation of motion is then; $$-c_1 v = m\frac{dv}{dt}$$

By integrating, $$t = -\int_{v_0}^{v} \frac{m\,dv}{c_1 v} = -\frac{m}{c_1}\ln\left(\frac{v}{v_0}\right)$$

Solving for v as a function of t gives; $$v = v_0 e^{-c_1 t/m}$$

A second integration gives $$x = \int_0^t v_0 e^{-c_1 t/m}\,dt$$

or $$x = \frac{m v_0}{c_1}\left(1 - e^{-c_1 t/m}\right)$$

Showing that after a long time ($t \sim \infty$) the block approaches a *limiting position* given by;

$$\boxed{x_{\lim} = m v_0 / c_1}$$

Horizontal Motion through a Fluid

Quadratic Resistance

(Exp.2.4.2)

The differential equation of motion in this case is;

$$-c_2 v^2 = m\frac{dv}{dt}$$

Similarly we can get v and the position x as a function of time.

Vertical Fall through a Fluid

1- Linear Case

For an object falling vertically in a resisting fluid, the force F_0 in this case, is the weight of the object, $-mg$. For the linear case of fluid resistance, the differential equation of motion is;

$$-mg - c_1 v = m\frac{dv}{dt}$$

Integrating and solving for v, we get

$$v = -\frac{mg}{c_1} + \left(\frac{mg}{c_1} + v_0\right)e^{-c_1 t/m}$$

Terminal Velocity

After a sufficient time ($t >> m/c_1$), the velocity approaches a limiting value ($-mg/c_1$). This limiting velocity of a falling body is called ***the terminal velocity*** (v_t). Hence *the terminal speed* is;

$$\boxed{v_t = \frac{mg}{c_1}}$$

The value of v_t/g *is known as* the ***characteristic time*** of the motion (τ). I.e ,

$$\boxed{\tau = \frac{v_t}{g} = \frac{m}{c_1}}$$

Note:

At the velocity v_t the force of resistance is just equal and opposite to the weight of the body so that the **net force** is **zero**, and so the **acceleration** is **zero**.

2- Quadratic case:

In this case *F(v)* $\propto v^2$ and the *differential equation* of motion is;

$$-mg - c_2 v^2 = m\frac{dv}{dt}$$

Similarly, ***the terminal speed*** *is ;*

$$\boxed{v_t = \sqrt{\frac{mg}{c_2}}}$$

And the ***characteristic time*** *is;*

$$\boxed{\tau = \frac{v_t}{g} = \sqrt{\frac{m}{c_2 g}}}$$

(Exp.2.4.3)

Linear or Quadratic ?

EXAMPLE 2.4.3

Falling Raindrops and Basketballs

Calculate the terminal speed in air and the characteristic time for (a) a very tiny spherical raindrop of diameter 0.1 mm = 10^{-4} m and (b) a basketball of diameter 0.25 m and mass 0.6 kg.

$$\boxed{\frac{Q}{L}=\frac{c_2 v|v|}{c_1 v}=\frac{0.22\,v|v|D^2}{1.55\times10^{-4}vD}=1.4\times10^3|v|D}$$

Raindrop

$$D=0.1mm=10^{-4}m$$

$$m=0.52\times10^{-9}kg$$

$$\boxed{\frac{Q}{L}=1.4\times10^3|v|10^{-4}=0.14v}$$

$$\frac{Q}{L}\geq1 \quad \text{if} \quad v\geq\frac{1}{0.14}=7.1\,m/s$$

Then, **linear case** should hold for the falling raindrop

Basketball

$$D=0.25m$$

$$m=0.6kg$$

$$\boxed{\frac{Q}{L}=1.4\times10^3|v|0.25=350v}$$

$$\frac{Q}{L}\geq1 \quad \text{if} \quad v\geq\frac{1}{350}=2.8\times10^{-3}\,m/s$$

Then, **Quadratic case** should hold for the falling Basketball

(Exp.2.4.3)

	Raindrop	Basketball
	$D = 0.1mm = 10^{-4} m$ $m = 0.52 \times 10^{-9} kg$	$D = 0.25m$ $m = 0.6kg$
	Then, **linear case** should hold for the falling raindrop	Then, **Quadratic case** should hold for the falling Basketball
The drag Coefficient	$c_1 = 1.55 \times 10^{-4}$ D $= 1.55 \times 10^{-8}$ **N.s/m**	$c_2 = 0.22$ D^2 $= 0.0138$ **N.s²/m²**
The terminal speed	$v_t = \frac{mg}{c_1} = 0.33 m/s$	$v_t = \sqrt{\frac{mg}{c_2}} = 20.6 m/s$
The characteristic time	$\tau = \frac{v_t}{g} = 0.034 s$	$\tau = \frac{v_t}{g} = 2.1 s$